

Non-Smooth Bifurcations of Uniformly Hyperbolic Invariant Manifolds in Skew Product Systems: Rigorous Results

Jordi-Lluís Figueras *

Dan Strängberg †

April 27, 2016

Abstract

In this paper we study the anti-integrable limit scenario of skew-product systems. We consider a generalization of such systems based on the Frenkel-Kontorova model, and prove the existence of orbits with any fibered rotation number in systems of both one and two degrees of freedom. In particular, our results also apply to two dimensional maps with degenerate potentials (vanishing second derivative), so extending the results of existence of Cantori for more general twist maps.

We also prove that under certain mild regularity conditions on the potential the structure of the orbits is of Cantor type. From our results we deduce the existence of the non-smooth folding bifurcation (conjectured by Figueras-Haro, *Different scenarios for hyperbolicity breakdown in quasiperiodic area preserving twist maps*, Chaos:25 (2015)).

Lastly we present a pair of results which are useful in determining if a potential satisfies the regularity conditions required for the Cantor sets of orbits to exist and are also of independent interest.

1 Skew product systems and their invariant sets

Given a compact manifold \mathcal{M} , a skew product system is a map $(h, F) : \mathcal{M} \times \mathbb{R}^n \rightarrow \mathcal{M} \times \mathbb{R}^n$ with $h(\theta, x) = h(\theta, y)$ for any $\theta \in \mathcal{M}$ and $x, y \in \mathbb{R}^n$. Of interest are the ones with quasiperiodic dynamics on $\mathcal{M} = \mathbb{T}^d$, $h(\theta) = \theta + \omega$, with $\omega \in \mathbb{R}^d$ being a totally irrational vector (if $m \cdot \omega = 0$ and $m \in \mathbb{Z}^d$ then $m \equiv 0$). Robust smooth invariant manifolds are *fiberwise hyperbolic*: the vector bundle $\mathcal{M} \times \mathbb{R}^n$ decomposes in an invariant continuous Whitney sum $E^s \oplus E^u$ such that there exists constants $C > 0$ and $0 < \lambda < 1$ satisfying

- $(\theta, v) \in E^s$ implies that $\|\pi_{\mathbb{R}^n}(h, D_2F)^k(\theta, v)\| \leq C\lambda^k\|v\|$ for $k \geq 0$;
- $(\theta, v) \in E^u$ implies that $\|\pi_{\mathbb{R}^n}(h, D_2F)^k(\theta, v)\| \leq C\lambda^k\|v\|$ for $k \leq 0$.

A Fiberwise Hyperbolic Invariant Manifold (FHIM) satisfies that it is the graph of a continuous function $K : \mathcal{M} \rightarrow \mathbb{R}^n$, $F(\theta, K(\theta)) = K(h(\theta))$, and persists under perturbations of F . See [10, 11, 15].

The question of possible bifurcation scenarios of FHIM is of great interest. A paradigm of this is the creation of Strange Non-Chaotic Attractors [1, 2, 4, 5, 7, 9, 12, 13, 17–24, 26–30]. These are one-parametric skew products on $\mathbb{T} \times \mathbb{R}$ satisfying that a smooth attracting invariant curve bifurcates to an only measurable attracting invariant curve with negative Lyapunov exponent. In all these cases $E^s = \mathbb{T} \times \mathbb{R}$ and $E^u = \emptyset$.

The case of non-attracting invariant curves, $E^u \neq \emptyset$, is more difficult to deal with. Numerical simulations are harder because they imply developing algorithms suitable for computing the FHIM and their bifurcations. Theoretical results must deal with the loss of regularity of the invariant manifold and with stable and unstable noncontinuous directions. Some results appear in the literature, see [11, 14, 16].

*Department of Mathematics, Uppsala University, Box 480, 75106 Uppsala (Sweden). figueras@math.uu.se.

†Department of Mathematics, Uppsala University, Box 480, 75106 Uppsala (Sweden). strangberg@math.uu.se.

In [11] the authors present a numerical study of some possible bifurcation scenarios of quasiperiodic invariant curves in the quasiperiodically driven standard map $(R_\omega, F) : \mathbb{T} \times \mathbb{R}^2 \rightarrow \mathbb{T} \times \mathbb{R}^2$ given by

$$\begin{cases} \bar{\theta} &= \theta + \omega \pmod{1} \\ \bar{x} &= x + \bar{y} \\ \bar{y} &= y - \frac{\partial W}{\partial x}(\theta, x) \end{cases} \quad (1)$$

where $\omega \in \mathbb{R} - \mathbb{Q}$ and $W \in C^1(\mathbb{T} \times \mathbb{R}, \mathbb{R})$. For the (γ, κ) -parametric family of $W(\theta, x) = \gamma x \sin(2\pi\theta) - \frac{\kappa}{(2\pi)^2} \cos(2\pi x)$, they observe three types of bifurcations: *smooth bifurcation*, *spiky breakdown*, and *folding breakdown*. In this paper we concentrate on the latter and a description of it goes as follows: There exists a critical value γ_c of the parameter γ such that:

- for all $\gamma < \gamma_c$ System (1) has an FHIM given as the graph of a smooth function $K_\gamma : \mathbb{T} \rightarrow \mathbb{R}^2$. These FHIM satisfy that they are uniformly hyperbolic and $\dim E^u = \dim E^s = 1$.
- At $\gamma = \gamma_c$ there exists $\theta_0 \in \mathbb{T}$ such that $\partial_\theta K_{\gamma_c}(\theta_0 + k\omega) = \infty$ for all $k \in \mathbb{Z}$.
- For $\gamma > \gamma_c$ there is a strange saddle: a bounded measurable invariant object with positive Lyapunov exponent.

The dynamical system (1) can be formulated in terms of the formal Lagrangian

$$L(\theta, \mathbf{x}) = \sum_{k \in \mathbb{Z}} \frac{1}{2} (x_{k+1} - x_k)^2 - W(\theta + k\omega, x_k). \quad (2)$$

Fixing $\theta_0 \in \mathbb{T}$ orbits of System (1) correspond to stationary solutions of the gradient flow of L ,

$$\begin{aligned} \dot{x}_k &= -\frac{\partial L}{\partial x_k}(\theta_0, \mathbf{x}) \\ &= x_{k+1} - 2x_k + x_{k-1} + \frac{\partial W}{\partial x}(\theta_0 + k\omega, x_k). \end{aligned} \quad (3)$$

The Lagrangian (2) is not unique, other possible formulations are possible. For example, in the case of System (1) a possible Lagrangian could be the classic Frenkel-Kontorova model with quasiperiodic spring lengths. However, all possible definitions define the same gradient flow (3).

In [11] a more general form of Equation (3) is considered, namely the anti-integrable limit scenario

$$\varepsilon(x_{k+1} - 2x_k + x_{k-1}) + V(\theta_0 + k\omega, x_k) = 0, \quad \forall k \in \mathbb{Z} \quad (4)$$

with $\varepsilon \approx 0$. Under the hypothesis of V satisfying that for any $\theta \in \mathbb{T}$ there is an $x \in \mathbb{R}$ such that $V(\theta, x) = 0$ and $\partial_x V(\theta, x) \neq 0$ a milder version of the folding breakdown is proven: There exists a one parametric family V_γ and $0 < a < b < 1$ such that for all $0 \leq \gamma \leq a$ the system (4) has an invariant FHIM, while for $b \leq \gamma \leq 1$ the system has an invariant strange saddle.

In this paper we completely prove the existence of the folding bifurcation and generalize it to more general systems.

Structure of the paper In Section 2 we present the main results in this paper. These results are divided into two different cases depending on the dimension of the systems. In Sections 3, 4 and 5 we present the proofs of all results stated in Section 2. Finally, in Section 6 we formulate some final results which are of independent interest and give additional information relating to the main results.

2 Formulation of the results

Let $h : \mathcal{M} \rightarrow \mathcal{M}$ be a homeomorphism of the compact space \mathcal{M} . We consider systems of the form

$$\varepsilon Z(\theta_k, x_{k+1}, x_k, x_{k-1}) + V(\theta_k, x_k) = 0, \quad \forall k \in \mathbb{Z} \quad (5)$$

where $Z \in C^r(\mathcal{M} \times \mathbb{R}^3, \mathbb{R})$, $V \in C^r(\mathcal{M} \times \mathbb{R}, \mathbb{R})$, $r \geq 1$, $\mathbf{x} = \{x_k\}_{k \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, $\{\theta_k\}_{k \in \mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$ with $\theta_k = h(\theta_{k-1})$ and $\varepsilon \in \mathbb{R}$. If $Z(\theta, a, b, c) = a + \tilde{Z}(\theta, b, c)$ and $\varepsilon \neq 0$ then System (5) defines a dynamical system on $\mathcal{M} \times \mathbb{R}^2$ given by

$$\begin{cases} \theta_{k+1} &= h(\theta_k) \\ x_{k+1} &= -\tilde{Z}(\theta_k, x_k, x_{k-1}) - \frac{1}{\varepsilon} V(\theta_k, x_k). \end{cases} \quad (6)$$

With a slight abuse of notation, we will call V the *potential*.

A key remark is that all the results that we present here are for values of ε small: the anti-integrable limit scenario. In the literature there are several results on this direction, see [3, 6, 25, 31]. In all these papers they deal with the case that the potential V does not vanish at the anti-integrable limit. In this paper we are able to provide proof of the existence of orbits even in the case that the derivative of the potential vanishes with even order: $V(\theta, x) \approx x^{2p+1}$, $p \in \mathbb{N}$.

The nature of the solutions of System (5) depends heavily on V . Our first result and its proof are an immediate generalization of a result appearing in [11] and are included for completeness.

Theorem 1. *Suppose that $Z \in C^r(\mathcal{M} \times \mathbb{R}^3, \mathbb{R})$, $V \in C^r(\mathcal{M} \times \mathbb{R}, \mathbb{R})$, $r \geq 1$, satisfies that $\partial_x V(\theta, x) \neq 0$ for all (θ, x) in a connected and bounded subset of the zero level set $V^{-1}(0)$. Then, there exists $\varepsilon_0 > 0$ and $K \in C^r((-\varepsilon_0, \varepsilon_0) \times \mathcal{M}, \mathbb{R})$ such that*

$$\varepsilon Z(\theta, K(\varepsilon, h(\theta)), K(\varepsilon, \theta), K(\varepsilon, h^{-1}(\theta))) + V(\theta, K(\varepsilon, \theta)) = 0 \quad (7)$$

holds for any $\theta \in \mathcal{M}$ and $|\varepsilon| < \varepsilon_0$. Moreover, in the case that Z defines a dynamical system as in (6) then it has positive Lyapunov exponents.

Notice that Theorem 1 implies that there exists bounded orbits lying on a smooth manifold: $x_k = K(\varepsilon, h^k(\theta_0))$.

The following results show that for more general V 's there are still bounded solutions. We present them for two different cases: the one dimensional and the two dimensional cases. We decided to present them separately because, although the statements are quite similar, the proofs are different. Moreover, this differentiation between cases is very natural, see Section 1. Note that the one dimensional case can be considered as a variation of the two dimensional case satisfying $\partial_c Z(\theta, a, b, c) \equiv 0$.

Remark 1. Throughout the rest of the paper the homeomorphism h is not needed and all results will remain true for general sequences $\{\theta_k\}_{k \in \mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$ with no modification of the proofs.

2.1 One dimensional case

Let $I = [-1, 1]$, $I_o = (-1, 1)$, and V be a potential satisfying

(C0) there is some $\varepsilon_0 > 0$ such that for every $|\varepsilon| \leq \varepsilon_0$ each connected component of the ε -level set of V is compactly contained in $\mathcal{M} \times I_o$ and projects surjectively onto \mathcal{M} .

Remark 2. For the results ε -level sets outside I can be allowed but restricting them avoids more cumbersome notation and makes the formulation of the results and their proofs easier.

We consider here a slight variation of Equation (5). Let $Z \in C^1(\mathcal{M} \times \mathbb{R}^2, \mathbb{R})$ satisfy $Z(\mathcal{M} \times I^2) \subset I_o$ and $\partial_x Z(\theta, x, y) \neq 0$ for every $(\theta, x, y) \in \mathcal{M} \times I^2$. We then consider the system

$$\varepsilon Z(\theta_k, x_{k+1}, x_k) + V(\theta_k, x_k) = 0, \quad \forall k \in \mathbb{Z}, \quad (8)$$

where $\theta_k = h(\theta_{k-1}) \in \mathcal{M}$. Our first result establishes the existence of solutions $\{x_k\} \in I^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z})$ of Equation (8).

Theorem 2. Suppose that $Z(\mathcal{M} \times I^2) \subset I_o$, $\partial_x Z(\theta, x, y) \neq 0$ for every $(\theta, x, y) \in \mathcal{M} \times I^2$, and $V \in C^1(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ satisfies that there are $-1 < t_0 < t_1 < 1$ and an $\varepsilon_0 > 0$ such that $V^{-1}([-\varepsilon_0, \varepsilon_0]) \subset \mathcal{M} \times [t_0, t_1]$ and projects surjectively onto \mathcal{M} . Let $h : \mathcal{M} \rightarrow \mathcal{M}$ be a map. Then for each $|\varepsilon| < \varepsilon_0$ and $\{\theta_k\}_{k \in \mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$, $\theta_k = h(\theta_{k-1})$, there is $\{x_k\}_{k \in \mathbb{Z}} \in I^{\mathbb{Z}}$ satisfying Equation (8).

Define the functions $f_\theta(x, y) = \varepsilon Z(\theta, x, y) + V(\theta, y)$. The hypothesis $\partial_x Z \neq 0$ implies that $f_\theta^{-1}(0)$ is a one dimensional submanifold of \mathbb{R}^2 for every $\theta \in \mathcal{M}$. If it is compact then by the classification of one dimensional manifolds each of its connected components must be diffeomorphic to either $[0, 1]$ or the circle \mathbb{T} . We give a special name to a certain class of such submanifolds which will be of special importance for us.

Definition 3. A connected component of $f_\theta^{-1}(0) \cap I^2$ is called *almost horizontal* if it is diffeomorphic to $[0, 1]$ with boundary points $p_1 \in \{-1\} \times I$ and $p_2 \in \{1\} \times I$ and if p_1, p_2 are its only points of intersection with the boundary of I^2 .

Definition 4. Given $\mathcal{M} \times \mathbb{R}^d$, the *fiber* of $\theta_0 \in \mathcal{M}$ is the set $\{\theta_0\} \times \mathbb{R}^d$.

Theorem 3. Under the same assumptions as in Theorem 2 and, in addition, assuming that for $|\varepsilon| > 0$ and $0 < \delta < 1$ there is $\{\theta_k\}_{k \in \mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$, $\theta_k = h(\theta_{k-1})$, satisfying the following:

1. The fiber over each θ_k contains an almost horizontal component with slope of absolute value at most $1 - \delta$ everywhere,
2. Infinitely many θ_k 's have fibers containing at least two almost horizontal components with slope of absolute value at most $1 - \delta$ everywhere.

Then for each $k \in \mathbb{Z}$ the coordinates x_k of all orbits of Equation (8) contained in almost horizontal components form a Cantor set.

2.2 Two dimensional case

In the two dimensional case we need that Z and V satisfy Condition (C0) and the following two:

(C1) $Z(\mathcal{M} \times I^3) \subset I_o$,

(C2) $\frac{\partial Z}{\partial x}(\theta, x, y, z) \neq 0 \neq \frac{\partial Z}{\partial z}(\theta, x, y, z)$ everywhere on $\mathcal{M} \times \mathbb{R}^3$.

Note that Condition (C1) is just a matter of scaling; since $\mathcal{M} \times I^3$ is compact any continuous function $f : \mathcal{M} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ can be multiplied by a constant $c > 0$ such that $cf(\mathcal{M} \times I^3) \subset I_o$. Note also that for $\varepsilon = 0$ Equation (5) reduces to the equation

$$V(\theta_k, x_k) = 0, \quad \forall k \in \mathbb{Z}$$

which, by Condition (C0), has a solution $\{x_k\}_{k \in \mathbb{Z}} \in I^{\mathbb{Z}}$ for any $\theta_0 \in \mathcal{M}$.

Theorem 4. Let $Z \in C^1(\mathcal{M} \times \mathbb{R}^3, \mathbb{R})$ and $V \in C^1(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ satisfying Conditions (C0), (C1) and (C2) and let $h : \mathcal{M} \rightarrow \mathcal{M}$ be a homeomorphism. Then for each $|\varepsilon| < \varepsilon_0$ and $\theta_0 \in \mathcal{M}$ there exists a sequence $\{x_k\}_{k \in \mathbb{Z}} \in I^{\mathbb{Z}}$ satisfying Equation (5).

As in the one dimensional case we introduce the functions $f_\theta(x, y, z) = \varepsilon Z(\theta, x, y, z) + V(\theta, y)$. By Condition (C2) the set $f_\theta^{-1}(0)$ is a two dimensional submanifold for each $\theta \in \mathcal{M}$. We again give a special name to a certain class of such submanifolds.

Definition 5. A connected component of $f_\theta^{-1}(0) \cap I^3$ is called *almost horizontal* if it projects surjectively onto $I^2 = \{(x, z)\}$, it is diffeomorphic to I^2 and if its boundary is entirely contained inside the boundary of I^3 and is the only intersection with the boundary of I^3 .

Furthermore each set $f_\theta^{-1}(0)$ is transversal to any $x = c$ or $z = c$ plane for any $c \in I$. This defines two foliations of $f_\theta^{-1}(0)$ that we call the *natural foliations*. The leaves of the natural foliations correspond to almost horizontal components of one dimensional problems.

Theorem 5. *Under the same conditions as in Theorem 4 and, in addition, assuming that for $|\varepsilon| > 0$ and $0 < \delta < 1$ there is $\{\theta_k\}_{k \in \mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$, $\theta_k = h(\theta_{k-1})$, satisfying the following:*

1. *The fiber over each θ_k contains an almost horizontal component such that each leaf of the natural foliations of the component has slope of absolute value at most $1 - \delta$ everywhere.*
2. *Infinitely many θ_k have fibers containing at least two almost horizontal components such that each leaf of the natural foliations of the components has slope of absolute value at most $1 - \delta$ everywhere.*

Then for each $k \in \mathbb{Z}$ the coordinates x_k of all orbits of Equation (5) contained in almost horizontal components form a Cantor set.

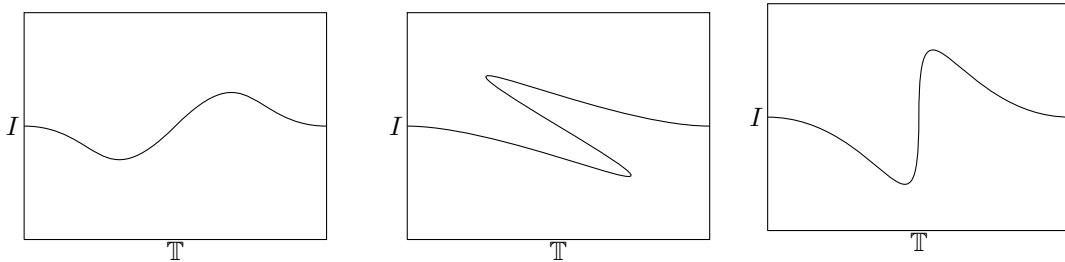
Remark 6. The same ideas that prove the existence of orbits for the 2-dimensional case generalize to the case $\varepsilon Z(\theta_k, x_{k+1}, x_k, x_{k-1}, \dots, x_{k-l}) + V(\theta_k, x_k)$ with appropriately modified assumptions on Z . The proofs for the structure of the orbit set also go through with the straightforward generalizations though the Cantor set disappears. Instead, we will have that the corresponding sets W_+ and W_- are essentially transversal $k - 1$ -dimensional subsets of the k -dimensional submanifold $f_{\theta_k}^{-1}(0)$. Thus the orbit set would, informally, have topological dimension $k - 2$. However, these would have a Cantor-like distribution in $f_{\theta_k}^{-1}(0)$.

2.3 Bifurcation diagram

By Theorem 1 the solution set of Equation (5) with a V satisfying $\partial_x V \neq 0$ in its 0-level set is a graph. On the other hand, Theorems 3 and 5 also show that all coordinates of certain types of solutions are contained in Cantor sets. For families of potentials V_t ranging from non-degenerate to those satisfying the conditions of Theorem 5, e.g. potentials with a folded 0-level set, the solution set of Equation (5) must undergo a bifurcation. In this section we discuss this bifurcation. We begin by making a definition.

Definition 7. The potential V is called *admissible* if for every $\theta \in \mathcal{M}$ there is a point $(\theta, y) \in V^{-1}(0)$ such that $\partial_y V(\theta, y) \neq 0$.

The set of admissible potentials contains the potentials with folded 0-level set, which are our prototypical example of a potential with a Cantor set of solutions to Equation (5) and are the subject of this discussion.



(a) A non-degenerate 0-level set of V for $\mathcal{M} = \mathbb{T}$. (b) An admissible folded 0-level set of V for $\mathcal{M} = \mathbb{T}$. (c) A nonadmissible 0-level set of V for $\mathcal{M} = \mathbb{T}$.

Figure 1

Note that any family of potentials going from non-degenerate to folded must pass through a nonadmissible potential as shown in Figure 1c. The bifurcation happens around this transition. The following explicit example of such a family is given for $\mathcal{M} = \mathbb{T}$ in [11]:

$$V_s(\theta, x) = (x^2 + a(\theta))(x - b(\theta)) + 2.15 - 0.15s$$

where $a(\theta) = 1.1 - 1.2 \sin(2\pi(\theta + 0.2))$ and $b(\theta) = 1.2 + 1.2 \cos^2(\pi\theta)$. For $s = 0$ the potential is admissible with a folded 0-level set and for $s = 1$ the potential is non-degenerate. As such the solution set to $\varepsilon Z + V_s = 0$ undergoes such a bifurcation as s goes from 1 to 0 for any Z satisfying the general conditions and any ε small enough.

Remark 8. In the case that Z defines a dynamical system we can talk about the stability of the orbits proven in the previous theorems. Following the results on stability in [11] we obtain that in the two-dimensional case the solutions on both the smooth manifold, Theorem 1, and on the Cantor set, Theorem 5, have positive Lyapunov exponent and one dimensional stable and unstable bundles. In the one-dimensional case, both theorems 1 and 3 imply that the solutions have positive Lyapunov exponent: they are repellers.

2.4 Fibered rotation numbers and commensurate and incommensurate Cantor Sets

The forward fibered rotation number of a sequence $\{x_k\}_{k \in \mathbb{Z}}$ is, if it exists, defined by

$$\rho = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=0}^{N-1} x_{k+1} - x_k.$$

The backward fibered rotation number is defined analogously by computing the limit for $N \rightarrow -\infty$. If both the forward and backward coincide, then it's called the fibered rotation number of the sequence.

By a slightly change on the proof of Theorems 2 or 4 we can prove the existence of orbits with any desired rotation number.

Corollary 1. *Under the assumptions of Theorem 2 or 4 and, if Z and V satisfy $Z(\theta, a, b, c) = Z(\theta, a + 1, b + 1, c + 1)$ and $V(\theta, x) = V(\theta, x + 1)$ for every $a, b, c, x \in \mathbb{R}$, and $Z(\mathcal{M} \times [-2, 2]^3) \subset [-1, 1]$, then for every $\omega \in \mathbb{R}$ System (5) admits a solution $\{y_k\}_{k \in \mathbb{Z}}$ with fibered rotation number ω and satisfying*

$$|y_k - k\omega| \leq 2. \quad (9)$$

The result in Corollary 1 goes in the same lines as the ones appearing in [25] but with a very remarkable difference: our results also apply for standard symplectic maps with potentials having vanishing derivatives. This could lead to the existence of this kind of sets with zero Lyapunov exponents, but we have not explored this possibility in this paper.

It is also worth noticing that Corollary 1 generalizes the results on the existence of Aubry-Mather sets appearing in [8]. It could be of interest to explore if with the variational techniques developed there similar results as Corollary 1 could be derived.

3 The non-degenerate case

Proof of Theorem 1. Consider the smooth functional $\mathcal{F} : C^r(\mathcal{M}, \mathbb{R}) \times \mathbb{R} \rightarrow C^r(\mathcal{M}, \mathbb{R})$ given by

$$\mathcal{F}(f, \varepsilon)(\theta) = \varepsilon Z(\theta, f(h(\theta)), f(\theta), f(h^{-1}(\theta))) + V(\theta, f(\theta)). \quad (10)$$

By hypothesis there is a function $f_0 \in C^r(\mathcal{M}, \mathbb{R})$ satisfying $\mathcal{F}(f_0, 0) \equiv 0$ with $D_1 \mathcal{F}(f_0, 0)$ invertible. Hence, by the Implicit Function Theorem there are neighborhoods U_1 of f_0 in $C^r(\mathcal{M}, \mathbb{R})$, U_2 of 0 in \mathbb{R} and a continuous map $\mathbb{R} \rightarrow C^r(\mathcal{M}, \mathbb{R})$ sending $\varepsilon \mapsto f_\varepsilon$ such that $\mathcal{F}(f, \varepsilon) \equiv 0$ if and only if $f = f_\varepsilon$ in these neighborhoods. Thus we can set $K(\varepsilon, \theta) = f_\varepsilon(\theta)$. \square

4 The degenerate case

We now consider the more general case where we do not put any conditions on $\partial_x V$.

4.1 One dimensional case

Proof of Theorem 2. The proof of the existence of the solutions $\{x_k\}_{k \in \mathbb{Z}}$ is essentially based on controlling the preimages of the associated dynamical system.

Let $k \in \mathbb{Z}$ and let some $x_{k+1} \in I$ be given. By the Intermediate Value Theorem equation

$$\varepsilon Z(\theta_k, x_{k+1}, x) + V(\theta_k, x) = 0$$

has a solution $x_k \in I$. Recursively, the equations

$$\varepsilon Z(\theta_{k-l}, x_{k-l+1}, x) + V(\theta_{k-l}, x) = 0$$

have solutions $x_{k-l} \in I$ for any $l > 0$.

To proceed, define the sets

$$B_k = \{\{x_m\}_{m \in \mathbb{Z}} \in I^{\mathbb{Z}} : \varepsilon Z(\theta_k, x_{k+1}, x_k) + V(\theta_k, x_k) = 0\}.$$

Note that B_k is a closed subset of $I^{\mathbb{Z}}$ for each k since $\varepsilon Z + V$ is a continuous function and hence its 0 level set is closed. The sequence of sets B_k satisfy the finite intersection property. To see this, note first that $B_k \cap B_l \neq \emptyset$ follows immediately for $|k-l| \geq 2$. To see that $B_k \cap B_{k-1} \neq \emptyset$ we can use the idea from above that given $x_{k+1}, x_k \in I$ satisfying $\varepsilon Z(\theta_k, x_{k+1}, x_k) + V(\theta_k, x_k) = 0$ we can always find $x_{k-1} \in I$ such that $\varepsilon Z(\theta_{k-1}, x_k, x_{k-1}) + V(\theta_{k-1}, x_{k-1}) = 0$. From this we can get a sequence $\{x_m\}_{m \in \mathbb{Z}} \in B_k \cap B_{k-1}$ by picking any $x_l \in I$ for $l \neq k+1, k, k-1$. By induction it follows that any intersection of the form

$$\bigcap_{k_1 \leq k \leq k_2} B_k$$

is also nonempty for $k_1 \leq k_2$. Since the intersection of any finite subcollection must contain such an intersection the finite intersection property follows.

Now since $I^{\mathbb{Z}}$ is compact by Tychonoff's theorem we get that the intersection

$$B = \bigcap_{k \in \mathbb{Z}} B_k$$

is nonempty and hence contains the solutions $\{x_k\}_{k \in \mathbb{Z}} \in I^{\mathbb{Z}}$ that we are looking for. \square

Proof of Theorem 3. Begin by fixing any $k \in \mathbb{Z}$. Let $(x_{k+1}, x_k) \in I^2$ be a solution of

$$f_{\theta_k}(x_{k+1}, x_k) = \varepsilon Z(x_{k+1}, x_k) + V(\theta_k, x_k) = 0$$

contained inside an almost horizontal component. Since $\partial_{x_{k+1}} f_{\theta_k} \neq 0$ we can use the implicit function theorem to find a surjective function $x_{k+1}(x_k)$ such that $f_{\theta_k}(x_{k+1}(x_k), x_k) = 0$ for every x_k in some closed set $O_1 \subset I$. Similarly, for $f_{\theta_{k+1}}^{-1}(0)$ we can also find an almost horizontal component and corresponding surjective function $x_{k+2}(x_{k+1}) : \tilde{O}_2 \rightarrow I$. By composition we thus get a surjective function $x_{k+2}(x_k) : O_2 \rightarrow I$ where $O_2 \subset O_1$. Continuing in this fashion we get a sequence of closed sets $\dots \subset O_n \subset O_{n-1} \subset \dots \subset O_2 \subset O_1 \subset I$ and corresponding functions $x_{k+n}(x_k) : O_n \rightarrow I$.

Now consider some fixed set O_n . If the fiber over θ_{n+1} has more than one almost horizontal component then there would be more than one choice of O_{n+1} , call them $O_{n+1, j_{n+1}}$ where j_{n+1} are indexed by a finite set $J_{n+1} \subset \mathbb{N}$. Thus O_n can be subdivided into $|J_{n+1}|$ connected components. Similarly, if the fiber over θ_{n+1} has only one almost horizontal component then $|J_{n+1}| = 1$.

Since by assumption there are infinitely many $n \in \mathbb{Z}$ such that $|J_n| \geq 2$ this allows for the construction of a Cantor set. Note that at level n the total number of connected components of points solving $f_{\theta_k} = 0, \dots, f_{\theta_{k+n}} = 0$ is $\prod_{i=1}^n |J_i| < \infty$. Denote the complete level n set by \overline{O}_n . Then \overline{O}_n is closed since it is a finite union of closed sets. It is compact since it is a subset of I and the sequence of sets $\{\overline{O}_n\}_{n \in \mathbb{N}}$ is nested such that $\overline{O}_{n+1} \subset \overline{O}_n$. By the finite intersection property it is therefore nonempty. Let $W = \bigcap_{n \geq 1} \overline{O}_n$. Then W is metrizable since it is a subset of a metric space. In order to show that it is a Cantor set it therefore remains to show that it has no isolated points and that it is totally disconnected.

Let $x \in W$ and N be an open neighborhood of x . At each level of the construction x belongs to some pulled back almost horizontal component. From the bound on the slope of the almost horizontal components we get that for any n

$$|O_n| \leq (1 - \delta)|O_{n-1}|$$

and by iteration we get that $|O_n| \leq 2(1 - \delta)^n$. Hence we have that for sufficiently large n there must be an entire component of the \overline{O}_n contained inside N . Furthermore, since there are infinitely many i for which θ_i has at least 2 surjective components this component contained inside N must eventually split into at least 2 components. Each of these components must contain points of W and therefore W cannot be connected and x cannot be isolated. \square

Remark 1. The construction of the sets O_n from the first paragraph of the above proof can also be used to show existence of solutions.

Before leaving the one dimensional case for the two dimensional case we prove the following one dimensional lemma that is useful for the two dimensional case.

Lemma 2. *Let $V \in C^r(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ satisfy Condition (C0) and let $Z \in C^r(\mathcal{M} \times \mathbb{R}^2, \mathbb{R})$, $r \geq 1$, satisfy*

- $Z(\mathcal{M} \times I^2) \subset I_o$,
- $\partial_x Z(\theta, x, y) \neq 0$ for every $(\theta, x, y) \in \mathcal{M} \times \mathbb{R}^2$.

Then for every $\theta \in \mathcal{M}$ and every $0 < |\varepsilon| < \varepsilon_0$ the set $f_\theta^{-1}(0) \cap I^2$ contains an almost horizontal component.

Proof. We begin by noting that 0 is a regular value of f_θ and so $f_\theta^{-1}(0)$ is a smooth one dimensional manifold and $f_\theta^{-1}(0) \cap I^2$ is compact. Furthermore, since $(x, y) \in f_\theta^{-1}(0) \cap I^2$ implies $(\theta, y) \in V^{-1}([-\varepsilon_0, \varepsilon_0]) \subset \mathcal{M} \times [t_0, t_1]$ we also have that each component of $f_\theta^{-1}(0) \cap I^2$ is in fact a smooth, compact submanifold of $I \times I_o$ contained in $I \times [t_0, t_1]$. By the classification of one dimensional, smooth, compact manifolds we then have that $f_\theta^{-1}(0) \cap I^2$ is diffeomorphic to a finite union of circles, line segments and isolated points. It must also project surjectively onto the x -axis since for any x Equation (8) can be solved for y by the intermediate value theorem. It remains to prove that the only possible case is the almost horizontal one.

First we note that no connected component of $f_\theta^{-1}(0)$ can be diffeomorphic to a circle since it cannot have any horizontal tangencies by the condition $\partial_x Z(\theta, x, y) \neq 0$.

Next we note that any isolated points or the endpoints of any component diffeomorphic to a line segment must be contained in $(\partial I) \times I_o$ by the fact that it must be contained inside $I \times [t_0, t_1]$ and that $f_\theta^{-1}(0)$ is a smooth one dimensional submanifold of \mathbb{R}^2 without boundary.

It also follows that $f_\theta^{-1}(0) \cap I^2$ must have at least one component which is diffeomorphic to a line segment. If this component intersects both the left boundary $\{-1\} \times I_o$ and the right boundary $\{1\} \times I_o$ we have an almost horizontal component and we are done. Otherwise, if both endpoints are contained in one side of the boundary, there must be another component diffeomorphic to a line segment which is either almost horizontal or whose endpoints are contained in the other side of the boundary in order for the set to project surjectively. In the first case we are again done. In the second case each of the curves divide the square I^2 into two parts: the inside, whose boundary is formed by the curve itself and the line segment connecting its endpoints, and the complementary outside. The values of f_θ on the inside and outside of such a curve differ by sign. This leads to a contradiction as on the part of I^2 outside both curves f_θ would have to take on values of both signs but never zero, see Figure 2 for an illustration in the case $\partial_x Z > 0$. Thus there must be an almost horizontal component even in this case. \square

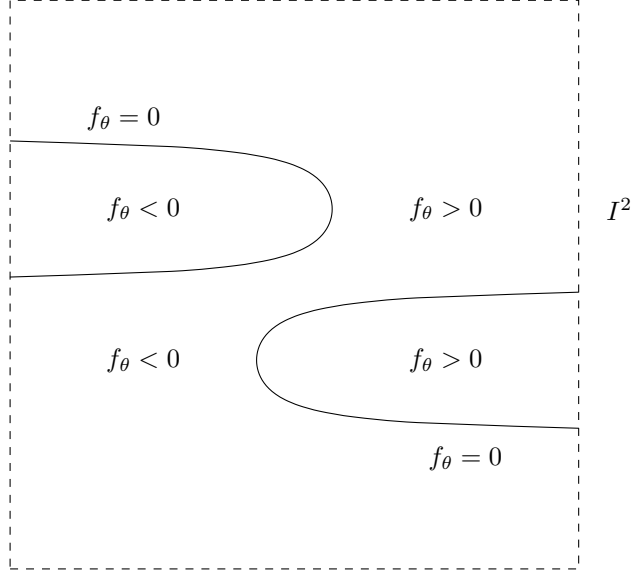


Figure 2: Impossibility of two components of $f_\theta^{-1}(0)$ intersecting $(\partial I) \times I_0$ on opposite sides with no almost horizontal component for the case $\partial_x Z > 0$. For $\partial_x Z < 0$ the signs are reversed.

Remark 3. In fact it can be shown that every connected component of the intersection of $V^{-1}([-\varepsilon_0, \varepsilon_0])$ with the fiber over $\theta \in M$ on which $V(\bullet, \theta)$ is surjective onto $[-\varepsilon_0, \varepsilon_0]$ must contain an almost horizontal component.

4.2 Two dimensional case

We suppose that Z and V are functions satisfying Conditions (C0), (C1) and (C2), from the introduction but will only consider $\varepsilon \neq 0$. We will prove the following.

Proof of Theorem 4. For a fixed $\theta_0 \in \mathcal{M}$ and $l > 0$ consider the finite dimensional system of equations

$$f_k(x_{k+1}, x_k, x_{k-1}) = \varepsilon Z(\theta_k, x_{k+1}, x_k, x_{k-1}) + V(\theta_k, x_k)$$

for $-l \leq k \leq l$, with $x_{l+1} = a \in I$ and $x_{-l-1} = b \in I$. Since

$$\partial_{x_{l-1}} f_l(a, x_l, x_{l-1}) \neq 0, \quad (11)$$

for any $x_{l-1} \in I$ we can find $x_l \in I$ such that $f_l(a, x_l, x_{l-1}) = 0$ by our general assumptions. By the implicit function theorem we can thus find open sets $U_l \subset I$ and $O_l \subset I$ along with a C^1 function $x_{l-1}(x_l) : U_l \rightarrow O_l$ such that

$$f_l(a, x_l, x_{l-1}(x_l)) = 0 \quad (12)$$

for any $x_l \in U_l$. Note that $f_l^{-1}(0) \cap I^2$ is a submanifold of $I_o \times I$, the first component corresponding to x_l and the second one to x_{l-1} . $f_l^{-1}(0)$ projects surjectively onto the x_{l-1} axis. In this setting we can apply Lemma 2 to show the existence of an *almost horizontal curve* in $\{(x_l, x_{l-1}) \in I^2 : f_l(a, x_l, x_{l-1}) = 0\}$ and we can therefore take $O_l = I$ and $x_{l-1}(x_l) : U_l \rightarrow I$ surjective. We use the word curve here to distinguish it from an almost horizontal component in the two dimensional case.

Next consider the equation $f_{l-1}(x_l, x_{l-1}(x_l), x_{l-2}) = 0$ with x_l restricted to U_l . By the intermediate value theorem we then have that for any x_{l-2} there is an x_l such that

$$f_{l-1}(x_l, x_{l-1}(x_l), x_{l-2}) = 0.$$

Using the implicit function theorem and Lemma 2 again we can find a closed, connected set $U_{l-1} \subset U_l$ and a surjective C^1 function $x_{l-2}(x_l) : U_{l-1} \rightarrow I$.

Now consider f_{l-2} with $x_{l-1}(x_l)$ and $x_{l-2}(x_l)$, and $x_l \in U_{l-1}$. Then for any x_{l-3} we can find, just as above, an $x_l \in U_{l-1}$ such that

$$f_{l-2}(x_{l-1}(x_l), x_{l-2}(x_l), x_{l-3}) = 0.$$

Thus we also find a closed, connected set $U_{l-2} \subset U_{l-1}$ and a C^1 surjection $x_{l-3}(x_l) : U_{l-2} \rightarrow I$.

Proceeding by induction we can find a nested sequence of closed set $U_{-l+1} \subset U_{-l+2} \subset \dots \subset U_{l-1} \subset U_l$ and corresponding C^1 surjections $x_k(x_l) : U_{k+1} \rightarrow I$. Lastly, consider $f_{-l}(x_{-l+1}, x_{-l}, b)$. By one final application of the intermediate value theorem we find an x_l such that

$$f_{-l}(x_{-l+1}(x_l), x_{-l}(x_l), b) = 0.$$

We have thus created an orbit of $f_k = 0$ for all $-l \leq k \leq l$ for any boundary conditions $x_{l+1} = a$, $x_{-l-1} = b$. Denote the set of all such orbits by $S_l^{a,b}$ and let $S_l = \bigcup_{a,b \in I} S_l^{a,b}$. As done in Theorem 3, this can be considered a closed subset of the compact space $I^{\mathbb{Z}}$. We then have $S_{l+1} \subset S_l$ so the sequence of closed sets $\{S_l\}_{l \in \mathbb{N}}$ is nested and, by above, each S_l is nonempty. Thus the sequence of sets has the finite intersection property and therefore the set $S_\infty = \bigcap_{l=0}^\infty S_l$ is nonempty and contains full orbits $\{x_k\}_{k \in \mathbb{Z}} \in I^{\mathbb{Z}}$ satisfying Equation (5). \square

Remark 4. Though the above proof only applies to the case $\varepsilon \neq 0$ it is easy to find solutions for $\varepsilon = 0$ as well since the x_k 's decouple, i.e. they are independent of each other, and by the general assumptions $V(\theta, \bullet)$ has at least one zero for every $\theta \in \mathcal{M}$. Furthermore these are contained in I .

Remark 5. There could be orbits which also pass through components which only intersect one boundary component of I^2 . We will not examine these in this paper.

Proof of Theorem 5. Fix $k \in \mathbb{Z}$. Using the implicit function theorem we can, as in the proof of Theorem 4, find a surjective function $x_{k-1}(x_{k+1}, x_k) : O_{-1} \rightarrow I$ corresponding to the almost horizontal component, where $O_{-1} \subset \{(x_k, x_{k+1}) \in I^2\}$ has surjective projection onto the x_{k+1} -axis, satisfying

$$f_{\theta_k}(x_{k+1}, x_k, x_{k-1}(x_{k+1}, x_k)) = 0$$

for every $(x_{k+1}, x_k) \in O_{-1}$. Similarly, for $f_{\theta_{k-1}}^{-1}(0)$ we can find a corresponding surjective function $x_{k-2}(x_k, x_{k-1}) : \tilde{O}_{-2} \rightarrow I$. Then, by considering the surjective map $g_{-1} : O_{-1} \rightarrow I^2$ given by

$$g_{-1}(x_{k+1}, x_k) = (x_k, x_{k-1}(x_{k+1}, x_k))$$

we can consider the pullback $O_{-2} := g_{-1}^{-1}(\tilde{O}_{-2})$. Continuing inductively we can construct surjective maps $x_{k-n}(x_{k-n+2}, x_{k-n+1}) : \tilde{O}_{-n} \rightarrow I$ and $g_{-n} : \tilde{O}_{-n} \rightarrow I^2$. By pulling back each \tilde{O}_{-n} through each g_{-n} in order we get sets $O_{-n} \subset O_{-n+1} \subset \dots \subset O_{-1}$. Note that each set in this sequence has surjective projection onto the x_{k+1} -axis.

In the same way we can also find a surjective function $x_{k+1}(x_k, x_{k-1}) : O_1 \rightarrow I$ satisfying

$$f_{\theta_k}(x_{k+1}(x_k, x_{k-1}), x_k, x_{k-1}) = 0$$

for every $(x_k, x_{k-1}) \in O_1$. We then consider the map $g_1 : O_1 \rightarrow I^2$ given by

$$g_1(x_k, x_{k-1}) = (x_{k+1}(x_k, x_{k-1}), x_k).$$

Thus we can consider the pullback $O_2 := g_1^{-1}(\tilde{O}_2)$. Proceeding inductively we again get a nested sequence of sets $O_n \subset O_{n-1} \subset \dots \subset O_1$, each set having surjective projection onto the x_{k-1} -axis.

Now consider some set O_n , $|n| \geq 1$. If the fiber over $\theta_{k+n\pm 1}$ has more than one surjective component then there would be more than one choice of $O_{n\pm 1}$, call them $O_{n\pm 1, j_{n\pm 1}}$ for $j_{n\pm 1} \in J_{n\pm 1}$ where $J_{n\pm 1}$ is a finite set. Thus O_n can be divided into $|J_{n\pm 1}|$ connected components. As in the 1-dimensional case we

will denote the complete level n set by \overline{O}_n . Letting $O_0 = I$ and $|J_0| = 1$ we can then write the number of connected components of \overline{O}_n as $\prod_{i=0}^n |J_i| < \infty$, each component projecting surjectively onto x_{k+1} if $n \leq -1$ and onto x_{k-1} if $n \geq 1$. They are also closed and nested.

For $n \geq 0$ we can thus consider the sets $\overline{W}_n^+ = \bigcap_{0 \leq i \leq n} \overline{O}_i$ and $\overline{W}_n^- = \bigcap_{-n \leq i \leq 0} \overline{O}_{-i}$ and their embeddings W_n^+, W_n^- into the almost horizontal component given by $W_n^+ = \{(x_{k+1}(x_k, x_{k-1}), x_k, x_k) : (x_k, x_{k-1}) \in \overline{W}_n^+\}$ and $W_n^- = \{(x_{k+1}, x_k, x_{k-1}(x_{k+1}, x_k)) : (x_{k+1}, x_k) \in \overline{W}_n^-\}$. Then each component of W_n^+ projects surjectively onto the x_{k-1} -axis while each component of W_n^- projects surjectively onto the x_{k+1} -axis. Since they are both contained inside a surface we must therefore have that each connected component of W_n^+ intersects every connected component of W_n^- and vice versa. Thus we define $W_n = W_n^+ \cap W_n^-$. Note that the sequence of sets W_n^+ and the sequence W_n^- are both nested and hence so is the sequence W_n . It follows by compactness that the set $W = \bigcap_{n \geq 0} W_n$ is nonempty. This is our prospective Cantor set. Note that since W is a subset of a metric space it is automatically metrizable so we only have to show that it has no isolated points and that it is totally disconnected. To this end, let $x = (x_{k+1}, x_k, x_{k-1}) \in W$ and N be a neighbourhood of x in \mathbb{R}^3 . Pick a sequence of sets O_n satisfying $x \in O_n$ for every $n \in \mathbb{Z}$. Then the sets O_n , as $n \rightarrow \pm\infty$, get contracted in the x_k -direction by a factor $1 - \delta$ at each level by the condition on the natural foliations. Therefore we can find a $n \geq 0$ large enough so that the x_k and x_{k-1} coordinates of the image of O_{-n} are contained inside N and such that the x_k and x_{k+1} coordinates of the image of O_n are also contained inside N . Thus the connected component of W_n corresponding to $O_n \cap O_{-n}$ is entirely contained inside N . Furthermore both O_n and O_{-n} must split into two or more connected components by assumption. Therefore the set N must contain points of W other than y and those points must be in a different connected component. Thus W can have no isolated points and is totally disconnected. \square

5 Existence of orbits with any fibered rotation number.

Proof of Corollary 1. We provide here the proof under the assumptions of Theorem 4. The other case is done mutatis mutandis.

Fix $\omega \in \mathbb{R}$ and let $m_k = [k\omega]$, where $[\alpha] = \max_{k \in \mathbb{Z}} \{k \leq \alpha\}$. Note that $m_{k+1} = m_k + \delta_k$ with $\delta_k \in \{0, 1, -1\}$. For any $a, b, c \in I$ the function Z satisfies that $Z(\theta, a + m_{k-1}, b + m_k, c + m_{k+1}) = Z(\theta, a, b, c) + G(\theta, m_{k-1}, m_k, m_{k+1})$ with $G : \mathcal{M} \times \mathbb{Z}^3 \rightarrow \mathbb{R}$ satisfying $|G| \leq 2$. Hence we obtain the equivalent System

$$\varepsilon \hat{Z}(\theta_k, x_{k+1}, x_k, x_{k-1}) + V(\theta_k, x_k) = 0, \quad \forall k \in \mathbb{Z}$$

with $\hat{Z}(\theta_k, a, b, c) = Z(\theta_k, a, b, c) + G(\theta_k, m_{k-1}, m_k, m_{k+1})$. By Theorem 4 it has a solution $\{x_k\}_{k \in \mathbb{Z}} \in I^{\mathbb{Z}}$. Finally, the sequence $\{y_k\}_{k \in \mathbb{Z}}$ defined by $y_k = x_k + m_k$ has fibered rotation number ω and satisfies Inequality (9) because of $|m_k - m_{k-1}| \leq 1$ for all $k \in \mathbb{Z}$. \square

6 Final remarks and further related results

We present here a pair of results which are of interest both independently and in relation to the main results.

Since Theorems 3 and 5 are formulated in terms of certain almost horizontal components it is of interest to know a priori if a certain system contain such components. These results give conditions on V that guarantee the existence of such almost horizontal components. In particular they show that admissible potentials satisfy the hypotheses of Theorems 3 and 5.

Proposition 1. *Let $(\theta_0, y_0) \in V^{-1}(0)$ such that $\partial_y V(\theta_0, y_0) \neq 0$. Then for every small enough ε there is a neighborhood of y_0 in I containing an almost horizontal component of $f_{\theta_0}^{-1}(0)$ which is also a graph over the x -axis. Furthermore, for each $0 < |\varepsilon| < \varepsilon_0$ the size of the projection onto the y -axis of each connected component is bounded from below by a positive constant.*

Proof. Consider the map $\mathcal{F} : C^1(I, \mathbb{R}) \times \mathbb{R} \rightarrow C^1(I, \mathbb{R})$ given by

$$\mathcal{F}(y, \varepsilon)(x) = \varepsilon Z(\theta_0, x, y(x)) + V(\theta_0, y(x)).$$

Then this map is Fréchet differentiable. Letting y_* denote the constant function $y_*(x) \equiv y_0$ we have $\mathcal{F}(y_*, 0) = 0$. Since $\partial_y V(\theta_0, y_*(x)) \neq 0$ we can apply the implicit function theorem giving us a family of functions $y_\varepsilon \in C^1(I)$ defined for sufficiently small $|\varepsilon| > 0$ such that $\varepsilon Z(\theta_0, x, y_\varepsilon(x)) + V(\theta_0, y_\varepsilon(x)) \equiv 0$. This proves the first part of the lemma.

For the second part we fix a small enough ε and consider the corresponding y_ε . By compactness we then have $|\partial_x Z(\theta_0, x, y)| \geq K_1$ for some constant $K_1 > 0$ and $|\partial_y Z(\theta_0, x, y) + \frac{1}{\varepsilon} \partial_y V(\theta_0, y)| \leq K_2$ for some constant $K_2 > 0$ for every $(x, y) \in I^2$. Now pick some (x_0, y_0) in an almost horizontal component. By the implicit function theorem we can then write $x(y)$ as a surjective function on some neighborhood of y_0 . From the above bounds we get that $|x'(y)| \leq \frac{K_2}{K_1}$ and hence the neighborhood around y_0 must have size at least $2\frac{K_1}{K_2}$. \square

Proposition 2. *Let $(\theta_0, y_0) \in V^{-1}(0)$ such that $\partial_y V(\theta_0, y_0) \neq 0$. Then for every small enough ε there is a neighborhood of y_0 in I containing an almost horizontal component of $f_{\theta_0}^{-1}(0)$ which is also a graph over the x - z plane. Furthermore, for each $0 < |\varepsilon| < \varepsilon_0$ the size of the projection onto the y -axis of each almost horizontal component is bounded from below by a positive constant.*

Proof. Follow the proof of Lemma 1 with $y = y(x, z)$ and $\mathcal{F} : C^1(I^2, \mathbb{R}) \times \mathbb{R} \rightarrow C^1(I^2, \mathbb{R})$.

For the second part, use the implicit function theorem to write $x = x(y, z)$ or $z = z(x, y)$. The bounds on the derivatives apply as before. \square

Acknowledgments

The authors want to thank professor de la Llave for his fruitful comments on the paper.

References

- [1] R. A. Adomaitis, I. G. Kevrekidis, and R. de la Llave. A computer-assisted study of global dynamic transitions for a noninvertible system. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 17(4):1305–1321, 2007.
- [2] L. Alsedà and M. Misiurewicz. Attractors for unimodal quasiperiodically forced maps. *J. Difference Equ. Appl.*, 14(10-11):1175–1196, 2008.
- [3] S. Aubry and G. Abramovici. Chaotic trajectories in the standard map. The concept of anti-integrability. *Phys. D*, 43(2-3):199–219, 1990.
- [4] Z. I. Bezhaeva and V. I. Oseledets. On an example of a “strange nonchaotic attractor”. *Funktsional. Anal. i Prilozhen.*, 30(4):1–9, 95, 1996.
- [5] K. Bjerklöv. SNA’s in the quasi-periodic quadratic family. *Comm. Math. Phys.*, 286(1):137–161, 2009.
- [6] Y.-C. Chen. Anti-integrability for the logistic maps. *Chin. Ann. Math. Ser. B*, 28(2):219–224, 2007.
- [7] S. Datta, R. Ramaswamy, and A. Prasad. Fractalization route to strange nonchaotic dynamics. *Phys. Rev. E*, 70(4):046203, Oct 2004.
- [8] R. de la Llave and E. Valdinoci. Ground states and critical points for generalized Frenkel-Kontorova models in \mathbb{Z}^d . *Nonlinearity*, 20(10):2409–2424, 2007.

- [9] U. Feudel, S. Kuznetsov, and A. Pikovsky. *Strange nonchaotic attractors*, volume 56 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. Dynamics between order and chaos in quasiperiodically forced systems.
- [10] J.-L. Figueras and A. Haro. Reliable computation of robust response tori on the verge of breakdown. *SIAM J. Appl. Dyn. Syst.*, 11:597–628, 2012.
- [11] J.-L. Figueras and À. Haro. Different scenarios for hyperbolicity in quasiperiodic area preserving twist maps. *Chaos*, 25, 2015.
- [12] P. Glendinning, U. Feudel, A. S. Pikovsky, and J. Stark. The structure of mode-locked regions in quasi-periodically forced circle maps. *Phys. D*, 140(3-4):227–243, 2000.
- [13] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke. Strange attractors that are not chaotic. *Phys. D*, 13(1-2):261–268, 1984.
- [14] A. Haro and R. de la Llave. Manifolds on the verge of a hyperbolicity breakdown. *Chaos*, 16(1):013120, 8, 2006.
- [15] A. Haro and R. de la Llave. A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: rigorous results. *J. Differential Equations*, 228(2):530–579, 2006.
- [16] A. Haro and R. de la Llave. A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: explorations and mechanisms for the breakdown of hyperbolicity. *SIAM J. Appl. Dyn. Syst.*, 6(1):142–207 (electronic), 2007.
- [17] A. Haro and J. Puig. Strange nonchaotic attractors in Harper maps. *Chaos*, 16(3):033127, 7, 2006.
- [18] A. Haro and C. Simó. To be or not to be a SNA: That is the question. Preprint, 2005.
- [19] T. H. Jäger. On the structure of strange non-chaotic attractors in pinched skew products. *Ergodic Theory Dynam. Systems*, 27(2):493–510, 2007.
- [20] T. H. Jäger. The creation of strange non-chaotic attractors in non-smooth saddle-node bifurcations. *Mem. Amer. Math. Soc.*, 201(945):vi+106, 2009.
- [21] T. H. Jäger. Strange non-chaotic attractors in quasiperiodically forced circle maps. *Comm. Math. Phys.*, 289(1):253–289, 2009.
- [22] K. Kaneko. Fractalization of torus. *Progr. Theoret. Phys.*, 71(5):1112–1115, 1984.
- [23] G. Keller. A note on strange nonchaotic attractors. *Fund. Math.*, 151(2):139–148, 1996.
- [24] S.-Y. Kim and W. Lim. Universal mechanism for the intermittent route to strange nonchaotic attractors in quasiperiodically forced systems. *Journal of Physics A: Mathematical and General*, 37(25):6477, 2004.
- [25] R. S. MacKay and J. D. Meiss. Cantori for symplectic maps near the anti-integrable limit. *Nonlinearity*, 5(1):149–160, 1992.
- [26] T. Nishikawa and K. Kaneko. Fractalization of a torus as a strange nonchaotic attractor. *Phys. Rev. E*, 54(6):6114–6124, Dec 1996.
- [27] H. M. Osinga and U. Feudel. Boundary crisis in quasiperiodically forced systems. *Phys. D*, 141(1-2):54–64, 2000.

- [28] H. M. Osinga, J. Wiersig, P. Glendinning, and U. Feudel. Multistability and nonsmooth bifurcations in the quasiperiodically forced circle map. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 11(12):3085–3105, 2001.
- [29] A. Prasad, S. S. Negi, and R. Ramaswamy. Strange nonchaotic attractors. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 11(2):291–309, 2001.
- [30] O. Sosnovtseva, U. Feudel, J. Kurths, and A. Pikovsky. Multiband strange nonchaotic attractors in quasiperiodically forced systems. *Physics Letters A*, 218(3-6):255 – 267, 1996.
- [31] J. J. P. Veerman and F. M. Tangerman. Intersection properties of invariant manifolds in certain twist maps. *Comm. Math. Phys.*, 139(2):245–265, 1991.